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On Separation Axioms in Fuzzifying Generalized Topology

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Abstract. In this paper we introduce and study the concept of fuzzifying separation axioms in fuzzifying generalized topological spaces.

1. introduction

Fuzzy topology as an important research field in fuzzy set theory has been developed into quite a mature discipline [1]–[7]. In contrast to classical topology, fuzzy topology is endowed with richer structure to a certain extent, which is manifested in different ways to generalize certain classical concepts. So far, according to [2], the kind of topologies defined by Chang [8] and Goguen [9] are called the topologies of fuzzy subsets, and further are naturally called L-topological spaces if a lattice L of membership values has been chosen. Loosely speaking, a topology of fuzzy subsets (resp. an L-topological space) is a family μ of fuzzy subsets (resp. L-fuzzy subsets) of nonempty set X, and μ satisfies the basic conditions of classical topologies. The concept of fuzzifying generalized topology was introduced and studied by the same authors [10]. All the results in this paper as a generalization of the results in [4], [11], [12] and [13]. That is, we introduce and study the concept of fuzzifying separation axioms in fuzzifying generalized topological spaces.

2. Preliminaries

First, we display the Lukasiewicz logic and corresponding set theoretical notations used in this paper. For any formula φ, the symbol $[\varphi]$ means the truth value of φ, where the set of truth values is the unit interval $[0, 1]$. We write $\models \varphi$ if $[\varphi] = 1$ for any interpretation. By $\models^w \varphi$(φ is feebly valid) we mean that for any valuation it always holds that $[\varphi] > 0$, and $\varphi \models^{ws} \xi$

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we mean that $[\varphi] > 0$ implies $[\xi]$=1. The truth valuation rules for primary fuzzy logical formulae and corresponding set theoretical notations are:

(1) (a) $[\lambda] = \lambda(\lambda \in [0,1])$;
  (b) $[\varphi \wedge \xi] = \min([\varphi], [\xi])$;
  (c) $[\varphi \to \xi] = \min(1, 1 - [\varphi] + [\xi])$;
(2) If $\tilde{A} \in \Im(X)$, then $[x \in \tilde{A}] := \tilde{A}(x)$.
(3) If X is the universe of discourse, then $[\forall x \varphi(x)] := \inf_{x \in X} [\varphi(x)]$.

In addition, the truth valuation rules for some derived formulae are

(1) $[\neg\varphi] := [\varphi \to 0] = 1 - [\varphi]$;
(2) $[\varphi \vee \xi] := [\neg(\neg\varphi \wedge \neg\xi)] = \max([\varphi], [\xi])$;
(3) $[\varphi \leftrightarrow \xi] := [(\varphi \to \xi) \wedge (\xi \to \varphi)]$;
(4) $[\varphi \underset{\circ}{\wedge} \xi] := [\neg(\varphi \to \neg\xi)] = \max(0, [\varphi] + [\xi] - 1)$;
(5) $[\varphi \vee \xi] := [\neg\varphi \to \xi] = \min(1, [\varphi] + [\xi])$;
(6) $[\exists x \varphi(x)] := [\neg\forall x \neg\varphi(x)] := \sup_{x \in X} [\varphi(x)]$;
(7) If $\tilde{A}, \tilde{B} \in \Im(X)$, then
  (a) $[\tilde{A} \subseteq \tilde{B}] := [\forall x (x \in \tilde{A} \to x \in \tilde{B})] = \inf_{x \in X} \min(1, 1 - \tilde{A}(x) + \tilde{B}(x))$;
  (b) $[\tilde{A} \equiv \tilde{B}] := [\tilde{A} \subseteq \tilde{B}] \wedge [\tilde{B} \subseteq \tilde{A}]$;
  (c) $[\tilde{A} \equiv \tilde{B}]$, where $\Im(X)$ is the family of all fuzzy sets in X.

Definition 1 ([10]). Let X be a universe of discourse, $\mu \in \Im(P(X))$ satisfying the following conditions:

(1) $\mu(x) = 1$, $\mu(\emptyset) = 1$;
(2) for any $\{A_\lambda : \lambda \in \Delta\}, \mu(\bigcup_{\lambda \in \Delta} A_\lambda) \geq \bigwedge_{\lambda \in \Delta} \mu(A_\lambda)$. Then μ is called a fuzzifying generalized topology and (X, μ) is a fuzzifying generalized topological space.

Definition 2 ([10]). The family of all fuzzifying generalized closed sets, denoted by $F \in \Im(P(X))$, is defined $A \in F := X - A \in \mu$, where $X - A$ is the complement of A.

Definition 3 ([10]). The fuzzifying generalized neighborhood system of a point $x \in X$ is denoted by $N_x \in \Im(P(X))$ and defined as follows:
$N_x(A) = \sup_{x \in B \subseteq A} \mu(B)$.

Definition 4 ([10]). The generalized closure $c_\mu(A)$ of A is defined as $c_\mu(A)(x) = 1 - N_x(X - A)$.

Definition 5 ([10]). For any $A \subseteq X$ the fuzzy set of the generalized interior points of A is called the interior of A, and given as follows: $i_\mu(A)(x) := N_x(A)$. Clearly, the definitions of $N_x(A)$ and $i_\mu(A)$ we have $\mu(A) = \inf_{x \in A} i_\mu(A)(x)$.

3. Fuzzifying μ-separation axioms

Remark 1. For simplicity we put the following notations:

$$K^{\mu}_{x,y} := \exists A((A \in N^{\mu}_x \wedge y \notin A) \vee (A \in N^{\mu}_y \wedge x \notin A)),$$
$$H^{\mu}_{x,y} := \exists B\exists C((B \in N^{\mu}_x \wedge y \notin B) \vee (C \in N^{\mu}_y \wedge x \notin C)),$$
$$M^{\mu}_{x,y} := \exists B\exists C(B \in N^{\mu}_x \wedge C \in N^{\mu}_y \wedge B \cap C = \emptyset),$$
$$V^{\mu}_{x,D} := \exists A\exists B(A \in N^{\mu}_x \wedge B \in \mu \wedge D \subseteq B \wedge A \cap B = \emptyset),$$
$$W^{\mu}_{A,B} := \exists G\exists H(G \in \mu \wedge H \in \mu \wedge A \subseteq G \wedge B \subseteq H \wedge G \cap H = \emptyset).$$

Definition 6. Let Ω be the class of all fuzzifying generalized topological spaces. The unary fuzzy predicates μ-$T_i \in \mathcal{I}(\Omega)$, i=0,1,2,3,4 and μ-$R_i \in \mathcal{I}(\Omega)$, i=0,1 are defined as follows:

$$(X, \mu) \in \mu\text{-}T_0 := \forall x\forall y(x \in X \wedge y \in X \wedge x \neq y) \to K^{\mu}_{x,y},$$
$$(X, \mu) \in \mu\text{-}T_1 := \forall x\forall y(x \in X \wedge y \in X \wedge x \neq y) \to H^{\mu}_{x,y},$$
$$(X, \mu) \in \mu\text{-}T_2 := \forall x\forall y(x \in X \wedge y \in X \wedge x \neq y) \to M^{\mu}_{x,y},$$
$$(X, \mu) \in \mu\text{-}T_3 := \forall x\forall D(x \in X \wedge D \in F_i \wedge x \notin D) \to V^{\mu}_{x,D},$$
$$(X, \mu) \in \mu\text{-}T_4 := \forall A\forall B(A \in F_i \wedge B \in F_i \wedge A \cap B = \emptyset) \to W^{\mu}_{A,B},$$
$$(X, \mu) \in \mu\text{-}R_0 := \forall x\forall y(x \in X \wedge y \in Y \wedge x \neq y) \to (K^{\mu}_{x,y} \to H^{\mu}_{x,y}),$$
$$(X, \mu) \in \mu\text{-}R_1 := \forall x\forall y(x \in X \wedge y \in Y \wedge x \neq y) \to (K^{\mu}_{x,y} \to M^{\mu}_{x,y}).$$

Lemma 1.

(1) $\models M^{\mu}_{x,y} \to H^{\mu}_{x,y}$,
(2) $\models H^{\mu}_{x,y} \to K^{\mu}_{x,y}$,
(3) $\models M^{\mu}_{x,y} \to K^{\mu}_{x,y}$.

Proof.

(1) Since $\{B, c\} \in P(X) : B \cap C = \emptyset\} \subseteq$ $\{B, c\} \in P(X) : y \notin B \wedge x \notin C\}$, then $M^{\mu}_{x,y} = \sup_{B\cap C=\emptyset} \min(N^{\mu}_x(B),$ $N^{\mu}_y(C) \leq \sup_{y\notin B,\, x\notin C} \min(N^{\mu}_x(B), N^{\mu}_y(C)) = H^{\mu}_{x,y}$.

(2) $[K^{\mu}_{x,y}] = \max\Big(\sup_{y\notin A} N^{\mu}_x(A), \sup_{x\notin B} N^{\mu}_y(A)\Big) \geq \sup_{y\notin A} N^{\mu}_x(A) \geq$ $\sup_{y\notin A,\, x\notin B} (N^{\mu}_x(A) \wedge N^{\mu}_x(b) = H^{\mu}_{x,y}$.

(3) From (1) and (2), it is obvious. □

Theorem 1.

(1) $\models (X, \mu) \in \mu\text{-}T_1 \to (X, \mu) \in \mu\text{-}T_0$,
(2) $\models (X, \mu) \in \mu\text{-}T_2 \to (X, \mu) \in \mu\text{-}T_1$.

Proof. The proof of (1) and (2) are obtained from Lemma 1 (2) and (1), respectively. □

Corollary 1. $\vDash (X,\mu) \in \mu\text{-}T_2 \to (X,\mu) \in \mu\text{-}T_0$.

Theorem 2. $\vDash (X,\mu) \in \mu\text{-}T_0 \leftrightarrow (\forall x \forall y (x \in X \wedge y \in X \wedge x \neq y \to (\neg(x \in c_\mu(\{y\}))) \vee \neg(y \in c_\mu(\{x\})))))$.

Proof.

$$\begin{aligned}[(X,\mu) \in \mu - T_0] &= \inf_{x\neq y} \max(\sup_{y\notin A} N_x^\mu(A), \sup_{x\notin A} N_y^\mu(A)) \\ &= \inf_{x\neq y} \max(N_x^\mu(X \sim \{y\}), N_y^\mu(X \sim \{x\})) \\ &= \inf_{x\neq y} \max(1 - c_\mu(\{y\})(x), 1 - c_\mu(\{x\})(y)) \\ &= \inf_{x\neq y} (\neg(c_\mu(\{y\})(x)) \vee \neg(c_\mu(\{x\})(y))) \\ &= \Big(\forall x \forall y \big(x \in X \wedge y \in X \wedge x \neq y \to \\ &\quad (\neg(x \in c_\mu(\{y\}))) \vee \neg(y \in c_\mu(\{x\})))\big)\Big).\end{aligned}$$

□

Theorem 3. *For any fuzzifying generalized topological space* $(X,\mu), \vDash (X,\mu) \in \mu\text{-}T_1 \leftrightarrow \forall x(\{x\} \in F_\mu)$.

Proof. For any $x_1, x_2, x_1 \neq x_2$,

$$\begin{aligned}[\forall x(\{x\} \in F_\mu)] &= \inf_{x\in X} F_\mu(\{x\}) \\ &= \inf_{x\in X} \mu(X \sim \{x\}) \\ &= \inf_{x\in X} \inf_{y\in X\sim\{x\}} N_y^\mu(X \sim \{x\}) \\ &\leq \inf_{y\in X\sim\{x_2\}} N_y^\mu(X \sim \{x_2\}) \\ &\leq N_{x_1}^\mu(X \sim \{x_2\}) = \sup_{x_2\notin A} N_{x_1}^\mu(A).\end{aligned}$$

Similarly, we have, $[\forall x(\{x\} \in F_\mu)] \leq \sup_{x_1\notin B} N_{x_2}^\mu(B)$. Then,

$$\begin{aligned}[\forall x(\{x\} \in F_\mu)] &= \inf_{x_1\neq x_2} \min(\sup_{x_2\notin A} N_{x_1}^\mu(A), \sup_{x_1\in B} N_{x_2}^\mu(B)) \\ &= \inf_{x_1\neq x_2} \sup_{x_1\notin B,\, x_2\notin A} \min(N_{x_1}^\mu(A), N_{x_2}^\mu(B)) \\ &= [(X,\mu) \in \mu - T_1].\end{aligned}$$

On the other hand,

$$\begin{aligned}
[(X,\mu)\in\mu-T_1] &= \inf_{x_1\neq x_2}\min(\sup_{x_2\notin A} N^{\mu}_{x_1}(A), \sup_{x_1\in B} N^{\mu}_{x_2}(B))\\
&= \inf_{x_1\neq x_2}\min(N^{\mu}_{x_1}(X\sim\{x_2\}), N^{\mu}_{x_2}(X\sim\{x_1\}))\\
&\leq \inf_{x_1\neq x_2} N^{\mu}_{x_1}(X\sim\{x_2\})\\
&= \inf_{x_2\in X}\inf_{x_1\in X\sim\{x_2\}} N^{\mu}_{x_1}(X\sim\{x_2\})\\
&= \inf_{x_2\in X}\mu(X\sim\{x_2\})\\
&= \inf_{x\in X}\mu(X\sim\{x\})\\
&= [\forall x(\{x\}\in F_{\mu})].
\end{aligned}$$

Thus, $[(X,\mu)\in\mu-T_1]=[\forall x(\{x\}\in F_{\mu})]$. □

Definition 7. The μ-local base $\mathcal{S}\beta_x$ of x is a function from $P(X)$ into I such that the following conditions are satisfied:

(1) $\vDash \mathcal{S}\beta_x\subseteq N^{\mu}_x$,
(2) $\vDash A\in N^{\mu}_x\to\exists B(B\in\mathcal{S}\beta_x\wedge x\in B\subseteq A)$.

Lemma 2. $\vDash A\in N^{\mu}_x\leftrightarrow\exists B(B\in\mathcal{S}\beta_x\wedge x\in B\subseteq A)$.

Proof. From condition (1) in Definition 7 we have $N^{\mu}_x(A)\geq N^{\mu}_x(B)\geq \mathcal{S}\beta_x(B)$ for each $B\in P(X)$ such that $x\in B\subseteq A$. So, $N^{\mu}_x(A)\geq\sup_{x\in B\subseteq A}\mathcal{S}\beta_x(B)$. From condition (2) in Definition 7, $N^{\mu}_x(A)\leq\sup_{x\in B\subseteq A}\mathcal{S}\beta_x(B)$. Hence $N^{\mu}_x(A)=\sup_{x\in B\subseteq A}\mathcal{S}\beta_x(B)$. □

Theorem 4. *If $\mathcal{S}\beta_x$ is a μ-local basis of x, then $\vDash (X,\mu)\in\mu\text{-}T_2\leftrightarrow \forall x\forall y(x\in X\wedge y\in X\wedge x\neq y\to\exists B(B\in\mathcal{S}\beta_x\wedge y\in\neg(c_{\mu}(B))))$.*

Proof. $[\forall x\forall y(x\in X\wedge y\in X\wedge x\neq y\to\exists B(B\in\mathcal{S}\beta_x\wedge y\in\neg(c_{\mu}(B))))]$

$$\begin{aligned}
&= \inf_{x\neq y}\sup_{B\in P(X)}\min(\mathcal{S}\beta_x(B), N^{\mu}_y(X\sim B))\\
&= \inf_{x\neq y}\sup_{B\in P(X)}\sup_{y\in C\subseteq X\sim B}\min(\mathcal{S}\beta_x(B),\mathcal{S}\beta_y(C))\\
&= \inf_{x\neq y}\sup_{B\cap C=\emptyset}\sup_{\substack{x\in D\subseteq B,\\ y\in E\subseteq C}}\min(\mathcal{S}\beta_x(D),\mathcal{S}\beta_y(E))\\
&= \inf_{x\neq y}\sup_{B\cap C=\emptyset}\min(\sup_{x\in D\subseteq B}\mathcal{S}\beta_x(D),\sup_{y\in E\subseteq C}\mathcal{S}\beta_y(E))\\
&= \inf_{x\neq y}\sup_{B\cap C=\emptyset}\min(N^{\mu}_x(B), N^{\mu}_y(C))\\
&= [(X,\mu)\in\mu-T_2].
\end{aligned}$$

□

Theorem 5. $\models (X,\mu) \in \mu\text{-}R_1 \to (X,\mu) \in \mu\text{-}R_0$.

Proof. From Lemma 1 (1) the proof is immediate. □

Theorem 6.

(1) $\models (X,\mu) \in \mu\text{-}T_1 \to (X,\mu) \in \mu\text{-}R_0$,
(2) $\models (X,\mu) \in \mu\text{-}T_1 \to (X,\mu) \in \mu\text{-}R_0 \wedge (X,\mu) \in \mu\text{-}T_0$,
(3) *If* $\mu\text{-}T_0(X,\mu) = 1$, *then*

$$\models (X,\mu) \in \mu\text{-}T_1 \leftrightarrow (X,\mu) \in \mu\text{-}R_0 \wedge (X,\mu) \in \mu\text{-}T_0.$$

Proof.

(1) We have,

$$\mu\text{-}R_1(X,\mu) = \inf_{x \neq y}[H^{\mu}_{x,y}] \leq \inf_{x \neq y}[K^{\mu}_{x,y} \to H^{\mu}_{x,y}] = \mu\text{-}R_0(X,\mu).$$

(2) It is obtained from (1) and from Theorem 1(1).

(3) Since $\mu\text{-}T_0(X,\mu) = 1$, for every $x, y \in X$ such that $x \neq y$ we have $[K^{\mu}_{x,y}] = 1$. Now,

$$\begin{aligned}[(X,\mu) \in \mu\text{-}R_0 \wedge (X,\mu) \in \mu\text{-}T_0] &= [(X,\mu) \in \text{-}\mu\text{-}R_0] \\ &= \inf_{x \neq y} \min(1, 1 - [K^{\mu}_{x,y}] + [H^{\mu}_{x,y}]) \\ &= \inf_{x \neq y}[H^{\mu}_{x,y}] = \text{-}\mu\text{-}T_1(X,\mu).\end{aligned}$$ □

Theorem 7.

(1) $\models (X,\mu) \in \mu\text{-}R_0 \wedge (X,\mu) \in \mu\text{-}T_0 \to (X,\mu) \in \mu\text{-}T_1$;
(2) *If* $\mu\text{-}T_0(X,\mu) = 1$, *then*

$$\models (X,\mu) \in \mu\text{-}R_0 \wedge (X,\mu) \in \mu\text{-}T_0 \leftrightarrow (X,\mu) \in \mu\text{-}T_1.$$

Proof.

(1) $[(X,\mu) \in \mu\text{-}R_0 \wedge (X,\mu) \in \mu\text{-}T_0]$

$$\begin{aligned}&= \max(0, \text{-}\mu\text{-}R_0(X,\mu) + \text{-}\mu\text{-}T_0(X,\mu) - 1) \\ &= \max(0, \inf_{x \neq y} \min(1, 1 - [K^{\mu}_{x,y}] + [H^{\mu}_{x,y}]) + \inf_{x \neq y}[K^{\mu}_{x,y}] - 1) \\ &\leq \max(0, \inf_{x \neq y} \min(1, 1 - [K^{\mu}_{x,y}] + [H^{\mu}_{x,y}]) + [K^{\mu}_{x,y}] - 1) \\ &= \inf_{x \neq y}[H^{\mu}_{x,y}] = \text{-}\mu\text{-}T_1(X,\mu).\end{aligned}$$

(2) $[(X,\mu) \in \mu\text{-}R_0 \wedge (X,\mu) \in \mu\text{-}T_0]$

$$\begin{aligned}&= [(X,\mu) \in \text{-}\mu\text{-}R_0] = \inf_{x \neq y} \min(1, 1 - [K^{\mu}_{x,y}] + [H^{\mu}_{x,y}]) \\ &= \inf_{x \neq y}[H^{\mu}_{x,y}] = \text{-}\mu\text{-}T_1(X,\mu),\end{aligned}$$

because μ-$T_0(X,\mu) = 1$, we have for each $x, y \in X$ such that $x \neq y$, $[K^{\mu}_{x,y}] = 1$. □

Theorem 8.

(1) $\vDash (X,\mu) \in \mu\text{-}T_0 \to ((X,\mu) \in \mu\text{-}R_0 \to (X,\mu) \in \mu\text{-}T_1)$.
(2) $\vDash (X,\mu) \in \mu\text{-}R_0 \to ((X,\mu) \in \mu\text{-}T_0 \to (X,\mu) \in \mu\text{-}T_1)$.

Proof. From Theorems 6 (1) and 7 (1), we have

(1) $[(X,\mu) \in \mu\text{-}T_0 \to ((X,\mu) \in \mu\text{-}R_0 \to (X,\mu) \in \mu\text{-}T_1)]$

$$\begin{aligned}
&= \min(1, 1 - [(X,\mu) \in \text{-}\mu\text{-}T_0] \\
&\quad + \min(1, 1 - [(X,\mu) \in \text{-}\mu\text{-}R_0] + [(X,\mu) \in \text{-}\mu\text{-}T_1])) \\
&= \min(1, 1 - [(X,\mu) \in \text{-}\mu\text{-}T_0] \\
&\quad + 1 - [(X,\mu) \in \text{-}\mu\text{-}R_0] + [(X,\mu) \in \text{-}\mu\text{-}T_1]) \\
&= \min(1, 1 - ([(X,\mu) \in \text{-}\mu\text{-}T_0] \\
&\quad + [(X,\mu) \in \text{-}\mu\text{-}R_0] - 1) + [(X,\mu) \in \text{-}\mu\text{-}T_1]) = 1.
\end{aligned}$$

(2) $[(X,\mu) \in \mu\text{-}R_0 \to ((X,\mu) \in \mu\text{-}T_0 \to (X,\mu) \in \mu\text{-}T_1)]$.

$$\begin{aligned}
&= \min(1, 1 - ([(X,\mu) \in \text{-}\mu\text{-}T_0] \\
&\quad + [(X,\mu) \in \text{-}\mu\text{-}R_0] - 1) + [(X,\mu) \in \text{-}\mu\text{-}T_1]) = 1.
\end{aligned}$$
□

Theorem 9.

(1) $\vDash (X,\mu) \in \mu\text{-}T_2 \to (X,\mu) \in \mu\text{-}R_1$,
(2) $\vDash (X,\mu) \in \mu\text{-}T_2 \to (X,\mu) \in \mu\text{-}R_1 \wedge (X,\mu) \in \mu\text{-}T_0$;
(3) *If* μ-$T_0(X,\mu) = 1$, *then* $\vDash (X,\mu) \in \mu\text{-}T_2 \leftrightarrow (X,\mu) \in \mu\text{-}R_1 \wedge (X,\mu) \in \mu\text{-}T_0$.

Proof.

(1) We have $\mu\text{-}T_2(X,\mu) = \inf_{x\neq y}[M^{\mu}_{x,y}] \leq \inf_{x\neq y}[K^{\mu}_{x,y} \to M^{\mu}_{x,y}] = \mu\text{-}R_1(X,\mu)$.
(2) It is obtained from (1) and Corollary 1.

(3) Since μ-$T_0(X,\mu) = 1$, then for each $x, y \in X$ such that $x \neq y$ we have $[K^{\mu}_{x,y}] = 1$.
Now,

$$\begin{aligned}
&[(X,\mu) \in \mu\text{-}R_1 \wedge (X,\mu) \in \mu\text{-}T_0] \\
&= [(X,\mu) \in \text{-}\mu\text{-}R_1] = \inf_{x\neq y} \min(1, 1 - [K^{\mu}_{x,y}] + [M^{\mu}_{x,y}]) \\
&= \inf_{x\neq y}[M^{\mu}_{x,y}] = \text{-}\mu\text{-}T_2(X,\mu).
\end{aligned}$$
□

Theorem 10.

(1) $\vDash (X,\mu) \in \mu\text{-}R_1 \wedge (X,\mu) \in \mu\text{-}T_0 \to (X,\mu) \in \mu\text{-}T_2$.
(2) $\vDash (X,\mu) \in \mu\text{-}R_1 \wedge (X,\mu) \in \mu\text{-}T_0 \leftrightarrow (X,\mu) \in \mu\text{-}T_2$.

Proof.

(1) $[(X,\mu) \in \mu\text{-}R_1 \wedge (X,\mu) \in \mu\text{-}T_0]$

$$= \max(0, \text{-}\mu\text{-}R_1(X,\mu) + \text{-}\mu\text{-}T_0(X,\mu) - 1)$$
$$= \max(0, \inf_{x \neq y} \min(1, 1 - [K^{\mu}_{x,y}] + [M^{\mu}_{x,y}]) + \inf_{x \neq y}[K^{\mu}_{x,y}] - 1)$$
$$\leq \max(0, \inf_{x \neq y} \min(1, 1 - [K^{\mu}_{x,y}] + [M^{\mu}_{x,y}]) + [K^{\mu}_{x,y}] - 1)$$
$$= \inf_{x \neq y}[M^{\mu}_{x,y}] = \text{-}\mu\text{-}T_2(X,\mu).$$

(2) $[(X,\mu) \in \mu\text{-}R_1 \wedge (X,\mu) \in \mu\text{-}T_0]$

$$= [(X,\mu) \in \text{-}\mu\text{-}R_1] = \inf_{x \neq y} \min(1, 1 - [K^{\mu}_{x,y}] + [M^{\mu}_{x,y}])$$
$$= \inf_{x \neq y}[M^{\mu}_{x,y}] = \text{-}\mu\text{-}T_2(X,\mu),$$

since $\mu\text{-}T_0(X,\mu) = 1$, then for each $x, y \in X$ such that $x \neq y, [K^{\mu}_{x,y}] = 1$. □

Theorem 11.

(1) $\models (X,\mu) \in \mu\text{-}T_0 \to ((X,\mu) \in \mu\text{-}R_1 \to (X,\mu) \in \mu\text{-}T_2)$.
(2) $\models (X,\mu) \in \mu\text{-}R_1 \to ((X,\mu) \in \mu\text{-}T_0 \to (X,\mu) \in \mu\text{-}T_2)$.

Proof.

(1) From Theorems 9(1) and 10(1) we have

$$[(X,\mu) \in \mu\text{-}T_0 \to ((X,\mu) \in \mu\text{-}R_1 \to (X,\mu) \in \mu\text{-}T_2)]$$
$$= \min(1, 1 - [(X,\mu) \in \text{-}\mu\text{-}T_0] + \min(1, 1 - [(X,\mu) \in \text{-}\mu\text{-}R_1] + [(X,\mu) \in \text{-}\mu\text{-}T_2]))$$
$$= \min(1, 1 - [(X,\mu) \in \text{-}\mu\text{-}T_0] + 1 - [(X,\mu) \in \text{-}\mu\text{-}R_1] + [(X,\mu) \in \text{-}\mu\text{-}T_2])$$
$$= \min(1, 1 - ([(X,\mu) \in \text{-}\mu\text{-}T_0] + [(X,\mu) \in \text{-}\mu\text{-}R_1] - 1) + [(X,\mu) \in \text{-}\mu\text{-}T_2]) = 1.$$

(2) The proof is similar to (1). □

Theorem 12. *If* $\mu\text{-}T_0(X,\mu) = 1$, *then*

(1) $\models ((X,\mu) \in \mu\text{-}T_0 \to ((X,\mu) \in \mu\text{-}R_0 \to (X,\mu) \in \mu\text{-}T_1))$
$\wedge((X,\mu) \in \mu\text{-}T_1 \to \neg((X,\mu) \in \mu\text{-}T_0 \to \neg((X,\mu) \in \mu\text{-}R_0)))$,
(2) $\models ((X,\mu) \in \mu\text{-}R_0 \to ((X,\mu) \in \mu\text{-}T_0 \to (X,\mu) \in \mu\text{-}T_1))$
$\wedge((X,\mu) \in \mu\text{-}T_1 \to \neg((X,\mu) \in \mu\text{-}T_0 \to \neg((X,\mu) \in \mu\text{-}R_0)))$,
(3) $\models ((X,\mu) \in \mu\text{-}T_0 \to ((X,\mu) \in \mu\text{-}R_0 \to (X,\mu) \in \mu\text{-}T_1))$
$\wedge((X,\mu) \in \mu\text{-}T_1 \to \neg((X,\mu) \in \mu\text{-}R_0 \to \neg((X,\mu) \in \mu\text{-}T_0)))$,
(4) $\models ((X,\mu) \in \mu\text{-}R_0 \to ((X,\mu) \in \mu\text{-}T_0 \to (X,\mu) \in \mu\text{-}T_1))$
$\wedge((X,\mu) \in \mu\text{-}T_1 \to \neg((X,\mu) \in \mu\text{-}R_0 \to \neg((X,\mu) \in \mu\text{-}T_0)))$.

Proof. For simplicity we put, $\mu\text{-}T_0(X,\mu) = \lambda$, $\mu\text{-}R_0(X,\mu) = \theta$ and $\mu\text{-}T_1(X,\mu) = \rho$. Now, applying Theorem 7(2), the proof is obtained with some relations in fuzzy logic as follows:

$$\begin{aligned}(1) = (\lambda \underset{\circ}{\wedge} \theta \leftrightarrow \rho) &= (\lambda \underset{\circ}{\wedge} \theta \to \rho) \wedge (\rho \to \lambda \underset{\circ}{\wedge} \theta)\\ &= (\lambda \underset{\circ}{\wedge} \theta \underset{\circ}{\wedge} \neg\rho) \wedge \neg(\rho \wedge \neg(\lambda \underset{\circ}{\wedge} \theta))\\ &= \neg(\lambda \underset{\circ}{\wedge} \neg(\neg(\theta \underset{\circ}{\wedge} \neg\rho))) \wedge \neg(\rho \underset{\circ}{\wedge}(\lambda \to \neg\theta))\\ &= (\lambda \to \neg(\theta \underset{\circ}{\wedge} \neg\rho)) \wedge (\rho \to \neg(\lambda \to \neg\theta))\\ &= (\lambda \to (\theta \to \rho)) \wedge (\rho \to \neg(\lambda \to \neg\theta)),\end{aligned}$$

since $\underset{\circ}{\wedge}$ is commutative one can have the proof of statements (2)-(4) in a similar way as (1). □

By a similar procedure to Theorem 11 one can have the following theorem.

Theorem 13.

(1) $\models ((X,\mu) \in \mu\text{-}T_0 \to ((X,\mu) \in \mu\text{-}R_1 \to (X,\mu) \in \mu\text{-}T_2))$
$\wedge((X,\mu) \in \mu\text{-}T_2 \to \neg((X,\mu) \in \mu\text{-}T_0 \to \neg((X,\mu) \in \mu\text{-}R_1))),$

(2) $\models ((X,\mu) \in \mu\text{-}R_1 \to ((X,\mu) \in \mu\text{-}T_0 \to (X,\mu) \in \mu\text{-}T_2))$
$\wedge((X,\mu) \in \mu\text{-}T_2 \to \neg((X,\mu) \in \mu\text{-}T_0 \to \neg((X,\mu) \in \mu\text{-}R_1))),$

(3) $\models ((X,\mu) \in \mu\text{-}T_0 \to ((X,\mu) \in \mu\text{-}R_1 \to (X,\mu) \in \mu\text{-}T_2))$
$\wedge((X,\mu) \in \mu\text{-}T_2 \to \neg((X,\mu) \in \mu\text{-}R_1 \to \neg((X,\mu) \in \mu\text{-}T_0))),$

(4) $\models ((X,\mu) \in \mu\text{-}R_1 \to ((X,\mu) \in \mu\text{-}T_0 \to (X,\mu) \in \mu\text{-}T_2))$
$\wedge((X,\mu) \in \mu\text{-}T_2 \to \neg((X,\mu) \in \mu\text{-}R_1 \to \neg((X,\mu) \in \mu\text{-}T_0))).$

Lemma 3.

(1) *If* $D \subseteq B$, *then* $\sup_{A\cap B=\emptyset} N_x^\mu(A) = \sup_{A\cap B=\emptyset, D\subseteq B} N_x^\mu(Z)$;

(2) $\sup_{A\cap B=\emptyset} \inf_{y\in D} N_y^\mu(X \sim A) = \sup_{\substack{A\cap B=\emptyset\\ D\subseteq B}} \mu(B)$.

Proof.

(1) Since $D \subseteq B$,

$$\sup_{A\cap B=\emptyset} N_x^\mu(A) = \sup_{A\cap B=\emptyset} N_x^\mu(A) \wedge [D \subseteq B] = \sup_{\substack{A\cap B=\emptyset\\ D\subseteq B}} N_x^\mu(A).$$

(2) Let $y \in D$ and $A \cap B = \emptyset$. Then,

$$\begin{aligned}\sup_{\substack{A\cap B=\emptyset\\ D\subseteq B}} \mu(B) &= \sup_{\substack{A\cap B=\emptyset\\ D\subseteq B}} \mu(B) \wedge [y \in D]\\ &= \sup_{y\in D\subseteq B\subseteq X\sim A} \mu(B)\\ &= \sup_{y\in D\subseteq X\sim A} \mu(B)\\ &= \sup_{A\cap B=\emptyset} \inf_{y\in D} N_y^\mu(X \sim A).\end{aligned}$$

□

Definition 8. $\mu\text{-}T_3^{(1)}(X,\mu) :=$

$$\forall x \forall D(x \in X \wedge D \in F_i \wedge x \notin D \to \exists A(A \in N_x^{\mu} \wedge (D \subseteq X \sim c_{\mu}(A)))).$$

Theorem 14. $\models (X,\mu) \in \mu\text{-}T_3 \leftrightarrow (X,1,2) \in \mu\text{-}T_3^{(1)}$.

Proof. Now,

$$\begin{aligned} \mu\text{-}T_3^{(1)}(X,\mu) &= \inf_{x\notin D} \min(1, 1 - i(X \sim D) \\ &\quad + \sup_{A\in P(X)} \min(N_x^{\mu}(A), \inf_{y\in D}(1 - c_{\mu}(A)(y)))), \\ &= \inf_{x\notin D} \min(1, 1 - i(X \sim D) \\ &\quad + \sup_{A\in P(X)} \min(N_x^{\mu}(A), \inf_{y\in D} N_y^{\mu}(X \sim A))), \end{aligned}$$

and

$$\mu\text{-}T_3(X,\mu) = \inf_{x\notin D} \min(1, 1 - i(X \sim D) + \sup_{\substack{A\cap B=\emptyset \\ D\subseteq B}} \min(N_x^{\mu}(A), \mu(B))).$$

So the result hold if we prove that

$$(\star) \qquad \sup_{A\in P(X)} \min(N_x^{\mu}(A), \inf_{y\in D} N_y^{\mu}(X \sim A)) = \sup_{\substack{A\cap B=\emptyset \\ D\subseteq B}} \min(N_x^{\mu}(A), \mu(B)).$$

It is clear that, on the left hand side of $(\star)$ when $A \cap C \neq \emptyset$, then there exists $y \in X$ such that $y \in D$ and $y \notin X \sim A$. So $\inf_{y\in D} N_y^{\mu}(X \sim A) = 0$ and thus $(\star)$ becomes

$$\sup_{\substack{A\in P(X) \\ A\cap B=\emptyset}} \min(N_x^{\mu}(A), \inf_{y\in D} N_y^{\mu}(X \sim A)) = \sup_{\substack{A\cap B=\emptyset \\ D\subseteq B}} \min(N_x^{\mu}(A), \mu(B)),$$

which is obtained from Lemma 3. □

Definition 9. $\mu\text{-}T_3^{(2)}(X,\mu) := \forall x \forall B(x \in B \wedge B \in \mu \to$

$$\exists A(A \in N_x^{\mu} \wedge c_{\mu}(A) \subseteq B)).$$

Theorem 15. $\models (X,\mu) \in \mu\text{-}T_3 \leftrightarrow (X,\mu) \in \mu\text{-}T_3^{(2)}$.

Proof. From Theorem 14, we have
$\mu\text{-}T_3(X,\mu)$
$= \inf_{x\notin D} \min(1, 1 - \mu(X \sim D) + \sup_{A\in P(X)} \min(N_x^{\mu}(A), \inf_{y\in D} N_y^{\mu}(X \sim A))).$

Now, if we put $B = X \sim D$, then

$$
\begin{aligned}
&\mu - T_3^{(2)}(X,\mu)\\
&= \inf_{x\in B} \min(1, 1 - i(B) + \sup_{A\in P(X)} \min(N_x^{\mu}(A), \inf_{y\in X\sim B} N_y^{\mu}(X\sim A)))\\
&= \inf_{x\notin D} \min(1, 1 - \mu(X\sim D) + \sup_{A\in P(X)} \min(N_x^{\mu}(A), \inf_{y\in D} N_y^{\mu}(X\sim A)))\\
&= \mu - T_3(X,\mu).
\end{aligned}
$$

□

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